

## ON A CONJECTURE OF MOREL

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ABSTRACT. In this note we prove that the  $\mathbb{A}^1$ -connected component sheaf  $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$  of an  $H$ -group  $\mathcal{X}$  is  $\mathbb{A}^1$ -invariant.

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## 1. INTRODUCTION

Let  $Sm/k$  denote the category of smooth, separated  $k$ -schemes and let  $PSh(Sm/k)$  denote the category of presheaves of sets on  $Sm/k$ . A functor  $\mathcal{X} : \Delta^{op} \rightarrow PSh(Sm/k)$  is called a simplicial presheaf or a space. Here  $\Delta$  is the category of simplices. Let  $\Delta^{op}PSh(Sm/k)$  denote the category of spaces.

$\Delta^{op}PSh(Sm/k)$  has a local model category structure with respect to the Nisnevich topology called the injective Nisnevich model structure. A morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a weak equivalence if the induced morphism on the Nisnevich stalks are weak equivalences of simplicial sets. Cofibrations are sectionwise injective morphisms and fibrations are defined using the right lifting property (see [4, 10]). The resulting homotopy category is denoted by  $\mathbf{H}_s(Sm/k)$ .

The Bousfield localisation of the local model structure on  $\Delta^{op}PSh(Sm/k)$  with respect to the class of maps  $\mathcal{X} \times \mathbb{A}^1 \rightarrow \mathcal{X}$  is called the  $\mathbb{A}^1$ -model structure (the  $\mathbb{A}^1$ -model structure for simplicial sheaves on  $Sm/k$  described in [10] was extended to simplicial presheaves in [5]). The resulting homotopy category is denoted by  $\mathbf{H}(k)$ .

For any space  $\mathcal{X}$ , define  $\pi_0^{\mathbb{A}^1}(\mathcal{X})$  to be the presheaf

$$U \in Sm/k \mapsto Hom_{\mathbf{H}(k)}(U, \mathcal{X}).$$

The presheaf  $\pi_0^{\mathbb{A}^1}(\mathcal{X})$  is homotopy invariant, i.e., for any  $U \in Sm/k$  the morphism  $\pi_0^{\mathbb{A}^1}(\mathcal{X})(U) \rightarrow \pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}_U^1)$ , induced by the projection  $\mathbb{A}_U^1 \rightarrow U$ , is bijective.

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Let  $a_{Nis} : PSh(Sm/k) \rightarrow Sh_{Nis}(Sm/k)$  denote the Nisnevich sheafification functor. The following conjecture of Morel states that the above property remains true after Nisnevich sheafification.

**Conjecture 1.1.** *For any  $U \in Sm/k$ , the morphism*

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(U) \rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}_U^1),$$

*induced by the projection  $\mathbb{A}_U^1 \rightarrow U$ , is bijective.*

In this paper, we prove the conjecture (theorem 3.15) for  $H$ -groups (definition 3.1) and homogeneous spaces on these (see definitions 3.3, 3.4).

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## 2. GENERALITIES ON THE NISNEVICH LOCAL MODEL STRUCTURE

In this section we briefly recall the Nisnevich Brown-Gersten property and give some consequences on the  $\pi_0$  functor.

Recall ([10, Definition 3.1.3]) that a cartesian square in  $Sm/k$

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X, \end{array}$$

is called an elementary distinguished square (in the Nisnevich topology), if  $p$  is an étale morphism and  $i$  is an open embedding such that  $p^{-1}(X - U) \rightarrow (X - U)$  is an isomorphism (endowing these closed subsets with the reduced subscheme structure).

A space  $\mathcal{X}$  is said to satisfy the Nisnevich Brown-Gersten property if for any elementary distinguished square in  $Sm/k$  as above, the induced square of simplicial sets

$$\begin{array}{ccc} \mathcal{X}(X) & \longrightarrow & \mathcal{X}(V) \\ \downarrow & & \downarrow \\ \mathcal{X}(U) & \longrightarrow & \mathcal{X}(W) \end{array}$$

is homotopy cartesian (see [10, Definition 3.1.13]).

Any fibrant space for the Nisnevich local model structure satisfies the Nisnevich Brown-Gersten property ([10, Remark 3.1.15]).

A space is  $\mathbb{A}^1$ -fibrant if and only if it is fibrant in the local model structure and  $\mathbb{A}^1$ -local ([10, Proposition 2.3.19]).

There exist endofunctors  $Ex$  (resp.  $Ex_{\mathbb{A}^1}$ ) of  $\Delta^{op}PSh(Sm/k)$  such that for any space  $\mathcal{X}$ , the object  $Ex(\mathcal{X})$  is fibrant (resp.  $Ex_{\mathbb{A}^1}\mathcal{X}$  is  $\mathbb{A}^1$ -fibrant). Moreover, there exists a natural morphism  $\mathcal{X} \rightarrow Ex(\mathcal{X})$  (resp.  $\mathcal{X} \rightarrow Ex_{\mathbb{A}^1}(\mathcal{X})$ ) which is a local weak equivalence (resp.  $\mathbb{A}^1$ -weak equivalence) ([10, Remark 3.2.5, Lemma 3.2.6, Theorem 2.1.66]).

**Remark 2.1.** *For the injective local model structure all spaces are cofibrant. Hence for any space  $\mathcal{X}$  and for any  $U \in Sm/k$ ,*

$$Hom_{\mathbf{H}_s(Sm/k)}(U, \mathcal{X}) = \pi_0(Ex(\mathcal{X})(U)).$$

Since  $Ex_{\mathbb{A}^1}(\mathcal{X})$  is  $\mathbb{A}^1$ -local,

$$Hom_{\mathbf{H}(k)}(U, \mathcal{X}) = Hom_{\mathbf{H}_s(Sm/k)}(U, Ex_{\mathbb{A}^1}(\mathcal{X})).$$

Moreover  $Ex_{\mathbb{A}^1}(\mathcal{X})$  is fibrant. Hence,

$$Hom_{\mathbf{H}(k)}(U, \mathcal{X}) = \pi_0(Ex_{\mathbb{A}^1}(\mathcal{X})(U)).$$

For any space  $\mathcal{X}$ , let  $\pi_0(\mathcal{X})$  be the presheaf defined by

$$U \in Sm/k \mapsto Hom_{\mathbf{H}_s(Sm/k)}(U, \mathcal{X}).$$

**Theorem 2.2.** *Let  $\mathcal{X}$  be a space. For any  $X \in Sm/k$ , such that  $\dim(X) \leq 1$ , the canonical morphism*

$$\pi_0(\mathcal{X})(X) \rightarrow a_{Nis}(\pi_0(\mathcal{X}))(X)$$

*is surjective.*

Before giving the proof we note the following consequence.

**Corollary 2.3.** *For any space  $\mathcal{X}$ , the canonical morphism*

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}_F^1) \rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}_F^1)$$

*is bijective for all finitely generated separable field extensions  $F/k$ .*

*Proof.* For any  $X \in Sm/k$ ,

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(X) = \pi_0(Ex_{\mathbb{A}^1}\mathcal{X})(X).$$

The canonical morphism

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}_F^1) \rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}_F^1)$$

is surjective (applying theorem 2.2 for the space  $Ex_{\mathbb{A}^1}(\mathcal{X})$ ). On the other hand, consider the following commutative diagram

$$\begin{array}{ccc} \pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}_F^1) & \xrightarrow{\quad} & \pi_0^{\mathbb{A}^1}(\mathcal{X})(F) \\ \downarrow & & \downarrow \wr \\ a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}_F^1) & \xrightarrow{\quad} & a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(F), \end{array}$$

where the horizontal morphisms are induced by the zero section  $F \xrightarrow{s_0} \mathbb{A}_F^1$ . The top horizontal morphism and the right vertical morphism are bijective. Hence the left vertical surjective morphism is injective.  $\square$

The proof of theorem 2.2 depends on the relation between homotopy pullback of spaces and pullback of the presheaves of connected components of those spaces.

Let  $I$  be a small category. There is a functor  $(I/-) : I \rightarrow Cat$  such that for any  $i \in I$ ,  $(I/-)(i) = I/i$ . Here  $Cat$  is the category of small categories and  $I/i$  is the over category. There is a functor  $N : Cat \rightarrow \Delta^{op}Sets$ , such that for any  $J \in Cat$ , the simplicial set  $N(J)$  is the nerve of the category  $J$ . Define  $N(I/-) := N \circ (I/-)$ .

A set  $S$  will be considered as a simplicial set in the obvious way : in every simplicial degree it is given by  $S$  and faces and degeneracies are identities. These simplicial sets are called discrete simplicial sets.

**Lemma 2.4.** *Let  $X : I \rightarrow \Delta^{op}Sets$  be a diagram of discrete simplicial sets. Then  $\lim_I X \cong \text{holim}_I X$ .*

*Proof.* By adjointness ([1, Ch. XI 3.3])

$$\mathrm{Hom}(\Delta^n \times N(I/-), X) = \mathrm{Hom}(\Delta^n, \mathrm{holim}_I X).$$

The functor  $\pi_0 : (\Delta^{op} \mathrm{Sets})^I \rightarrow (\mathrm{Sets})^I$  is left adjoint to the functor  $N : (\mathrm{Sets})^I \rightarrow (\Delta^{op} \mathrm{Sets})^I$ , where  $N$  maps a diagram of sets to the same diagram of discrete simplicial sets. Hence  $\mathrm{Hom}(\Delta^n \times N(I/-), X) = \mathrm{Hom}(\bullet_I, X)$ , where  $\bullet_I$  is the diagram of sets given by the one element set for each  $i \in I$ . But  $\mathrm{Hom}(\bullet_I, X) = \mathrm{Hom}(\bullet, \mathrm{lim}_I X)$ , by adjointness. Therefore, we get our result.  $\square$

**Remark 2.5.** Let  $X : I \rightarrow \Delta^{op} \mathrm{Sets}$  be a diagram such that each  $X(i)$  is fibrant for all  $i \in I$ . The canonical morphism  $X(i) \rightarrow \pi_0(X(i))$  induces a morphism  $\mathrm{holim}_I(X) \rightarrow \mathrm{lim}_I \pi_0(X)$ . This gives the following morphism

$$(2.1) \quad \pi_0(\mathrm{holim}_I(X)) \rightarrow \mathrm{lim}_I \pi_0(X).$$

**Lemma 2.6.** Suppose that  $I$  is the pullback category  $1 \rightarrow 0 \leftarrow 2$  and let  $D : I \rightarrow \Delta^{op} \mathrm{Sets}$  be a digram  $X \xrightarrow{p} Y \xleftarrow{q} Z$  such that  $X, Y, Z$  are fibrant. Then the map (2.1) is surjective.

*Proof.* By [1, Ch. XI 4.1.(iv), 5.6]  $\mathrm{holim}_I(X) \cong X' \times_Y Z$ , where  $X \rightarrow X' \xrightarrow{p'} Y$  is a factorisation of  $p$  into a trivial cofibration followed by a fibration  $p'$ . Since  $\pi_0(X) \cong \pi_0(X')$ , it is enough to show that

$$\pi_0(X' \times_Y Z) \rightarrow \pi_0(X') \times_{\pi_0(Y)} \pi_0(Z)$$

is surjective. So we can assume that  $p$  is a fibration. Let  $s \in \pi_0(X) \times_{\pi_0(Y)} \pi_0(Z)$ .  $s$  can be represented (not uniquely) by  $(x, y, z)$ , where  $(x, z) \in X_0 \times Z_0$  and  $y \in Y_1$  such that  $d_0(y) = p(x)$  and  $d_1(y) = q(z)$ . Since  $p$  is a fibration, we can lift the path  $y$  to a path  $y' \in X_1$  such that  $d_0(y') = x$  and  $x' := d_1(y')$  maps to  $q(z)$ .  $\mathrm{holim}_I D \cong X \times_Y Z$ . Therefore  $(x', z) \in \mathrm{holim}_I D$  which maps to  $s$ . This proves the surjectivity.  $\square$

**Remark 2.7.** Under the condition of lemma 2.6, the map (2.1) may not be injective. Indeed, if  $Y$  is connected,  $X$  is the universal cover of  $Y$  and  $Z = \bullet$ , then (2.1) is injective if and only if  $Y$  is simply connected.

A noetherian  $k$ -scheme  $X$ , which is the inverse limit of a left filtering system  $(X_\alpha)_\alpha$  with each transition morphism  $X_\beta \rightarrow X_\alpha$  being an étale affine morphism between smooth  $k$ -schemes, is called an essentially smooth  $k$ -scheme. For any  $X \in \mathrm{Sm}/k$  and any  $x \in X$ , the local schemes  $\mathrm{Spec}(O_{X,x})$  and  $\mathrm{Spec}(O_{X,x}^h)$  are essentially smooth  $k$ -schemes.

**Lemma 2.8.** Let  $\mathcal{X}$  be a space. For any essentially smooth discrete valuation ring  $R$ , the canonical morphism

$$\pi_0(\mathcal{X})(R) \rightarrow a_{Nis}(\pi_0(\mathcal{X}))(R)$$

is surjective.

*Proof.* By remark 2.1 we can assume that  $\mathcal{X}$  is fibrant.

Let  $F = \mathrm{Frac}(R)$  and let  $R^h$  be the henselisation of  $R$  at its maximal ideal. Suppose  $s \in a_{Nis}(\pi_0(\mathcal{X}))(R)$ . Then for the image of  $s$  in  $a_{Nis}(\pi_0(\mathcal{X}))(R^h)$ , there exists a Nisnevich neighbourhood of the closed point  $p : W \rightarrow \mathrm{Spec}(R)$  and  $s' \in \pi_0(\mathcal{X})(W)$ , such that  $s'$  gets mapped to  $s|_W \in a_{Nis}(\pi_0(\mathcal{X}))(W)$ . Let  $L = \mathrm{Frac}(W)$ . For any finitely generated separable field extension  $F/k$ , the

map  $\pi_0(\mathcal{X})(F) \rightarrow a_{Nis}(\pi_0(\mathcal{X}))(F)$  is bijective. Hence,  $s'|_L$  is same as  $s|_L$ . We get two sections  $s' \in \pi_0(\mathcal{X})(W)$  and  $s|_F \in \pi_0(\mathcal{X})(F)$ , such that  $s'|_L = s|_L$ . By lemma 2.6 and the fact that  $\mathcal{X}$  satisfies the Nisnevich Brown-Gersten property, we find an element  $s_v \in \pi_0(\mathcal{X})(R)$  which gets mapped to  $s$ . Therefore,  $\pi_0(\mathcal{X})(R) \rightarrow a_{Nis}(\pi_0(\mathcal{X}))(R)$  is surjective.  $\square$

*Proof of theorem 2.2.* Let  $X \in Sm/k$  and  $\dim(X) = 1$ . Let  $\alpha$  be an element of  $a_{Nis}(\pi_0(\mathcal{X}))(X)$ . This  $\alpha$  gives  $\alpha_p \in a_{Nis}(\pi_0(\mathcal{X}))(O_{X,p})$  for every codimension 1 point  $p \in X$ , such that  $\alpha_p|_{K(X)} = \alpha_q|_{K(X)}$ , for all  $p, q \in X^{(1)}$ . By the surjectivity of

$$\pi_0(\mathcal{X})(O_{X,p}) \rightarrow a_{Nis}(\pi_0(\mathcal{X}))(O_{X,p})$$

and bijectivity of

$$\pi_0(\mathcal{X})(K(X)) \rightarrow a_{Nis}(\pi_0(\mathcal{X}))(K(X)),$$

we get elements  $\alpha'_p \in \pi_0(\mathcal{X})(O_{X,p})$  mapping to  $\alpha_p$ , such that  $\alpha'_p|_{K(X)} = \alpha'_q|_{K(X)}$  for  $p, q \in X^{(1)}$ .

Fix a  $p \in X^{(1)}$ . There exists an open set  $U$  and  $\beta \in \pi_0(\mathcal{X})(U)$ , such that  $\beta|_{O_{X,p}} = \alpha'_p$ . Let  $\beta' \in a_{Nis}(\pi_0(\mathcal{X})(U))$  be the image of  $\beta$ . Suppose that  $\beta' \neq \alpha|_U$ , but  $\beta'|_{O_{X,p}} = \alpha_p$ . Hence there exists  $U' \subset U$ , such that  $\beta'|_{U'} = \alpha|_{U'}$ .

So we can assume by Noetherian property of  $X$  that there exists a maximal open set  $U \subset X$  and  $\alpha' \in \pi_0(\mathcal{X})(U)$ , such that  $\alpha'$  gets mapped to  $\alpha|_U$ . If  $U \neq X$ , then there exists a codimension one point  $q \in X \setminus U$ . We can get an open neighborhood  $U_q$  and an element  $\alpha'' \in \pi_0(\mathcal{X})(U_q)$ , such that  $\alpha''$  gets mapped to  $\alpha|_{U_q}$ . But by construction of these  $\alpha'', \alpha'$  we know that  $\alpha''|_{K(X)} = \alpha'|_{K(X)}$ . Hence there exists an open set  $U' \subset U_q \cap U$ , such that  $\alpha''|_{U'} = \alpha'|_{U'}$ . Let  $Z = U_q \cap U \setminus U'$ . Since  $\dim(X) = 1$ , the set  $Z$  is finite collection of closed points. Therefore,  $Z$  is closed in  $U$ . Let  $U'' = U \setminus Z$  be the open subset of  $U$ . Note that  $U'' \cap U_q = U'$ . Denote  $U'' \cup U_q = U \cup U_q$  by  $V$ .

Let  $\alpha'|_{U''} \in \pi_0(\mathcal{X})(U'')$  be the restriction of  $\alpha'$  to  $U''$ . Hence,  $\alpha'|_{U''}$  gets mapped to  $\alpha|_{U''}$  and  $\alpha'|_{U''}$  restricted to  $U'$  is same as  $\alpha''$  restricted to  $U'$ . As  $\mathcal{X}$  is Nisnevich fibrant, it satisfies the Zariski Brown-Gersten property. By lemma 2.6, we get a section  $s_V \in \pi_0(\mathcal{X})(V)$  which gets mapped to  $s|_V$ . This gives a contradiction to the maximality of  $U$ . This finishes the proof of the theorem.  $\square$

### 3. $H$ -GROUPS AND HOMOGENEOUS SPACES

In this section we prove  $\mathbb{A}^1$ -invariance of  $a_{Nis}(\pi_0^{\mathbb{A}^1})$  for  $H$ -groups and homogeneous spaces for  $H$ -groups.

**Definition 3.1.** Let  $\mathcal{X}$  be a pointed space, i.e.,  $\mathcal{X}$  is a space endowed with a morphism  $x : \text{Spec}(k) \rightarrow \mathcal{X}$ . It is called an  $H$ -space if there exists a base point preserving morphism  $\mu : (\mathcal{X} \times \mathcal{X}) \rightarrow \mathcal{X}$ , such that  $\mu \circ (x \times id_{\mathcal{X}})$  and  $\mu \circ (id_{\mathcal{X}} \times x)$  are equal to  $id_{\mathcal{X}}$  in  $\mathbf{H}(k)$ . Here  $\mathcal{X} \times \mathcal{X}$  is pointed by  $(x, x)$ . It is called an  $H$ -group if :

- (1)  $\mu \circ (\mu \times id_{\mathcal{X}})$  is equal to  $\mu \circ (id_{\mathcal{X}} \times \mu)$  in  $\mathbf{H}(k)$  modulo the canonical isomorphism  $\alpha : \mathcal{X} \times (\mathcal{X} \times \mathcal{X}) \rightarrow (\mathcal{X} \times \mathcal{X}) \times \mathcal{X}$ .
- (2) There exists a morphism  $(.)^* : \mathcal{X} \rightarrow \mathcal{X}$ , such that  $\mu \circ (id_{\mathcal{X}}, (.)^*)$  and  $\mu \circ ((.)^*, id_{\mathcal{X}})$  are equal to the constant map  $c : \mathcal{X} \rightarrow \mathcal{X}$  in  $\mathbf{H}(k)$ . Here the image of the constant map  $c$  is  $x$ .

**Remark 3.2.** Recall from [10, 3.2.1] that

$$Ex_{\mathbb{A}^1} = Ex^{\mathcal{G}} \circ (Ex^{\mathcal{G}} \circ Sing_{*}^{\mathbb{A}^1})^{\mathbb{N}} \circ Ex^{\mathcal{G}}.$$

The functors  $Ex^G$  and  $Sing_*^{\mathbb{A}^1}$  commutes with finite limits by [10, 2.3.2, Theorem 2.1.66]. Also filtered colimit commutes with finite products. Therefore,  $Ex_{\mathbb{A}^1}$  commutes with finite products. If  $\mathcal{X}$  is an  $H$ -group as described in 3.1, then the morphisms  $Ex_{\mathbb{A}^1}(x)$ ,  $Ex_{\mathbb{A}^1}(\mu)$  and  $Ex_{\mathbb{A}^1}((\cdot)^*)$  satisfy the conditions of the definition 3.1. Hence,  $Ex_{\mathbb{A}^1}(\mathcal{X})$  is also an  $H$ -group.

Suppose that  $a, b, c \in \pi_0(Ex_{\mathbb{A}^1}(\mathcal{X}))(U)$  for some  $U \in Sm/k$ . Let  $f, g : \mathcal{Y} \rightarrow \mathcal{Z}$  be morphisms between  $\mathbb{A}^1$ -fibrant spaces such that  $f$  is equal to  $g$  in  $\mathbf{H}(k)$ , then  $f$  and  $g$  are simplicially homotopic. Using this, we get  $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$ ,  $\mu(a, x) = a = \mu(x, a)$  and  $\mu(a, a^*) = \mu(a^*, a) = x$ . Hence,  $\pi_0(Ex_{\mathbb{A}^1}(\mathcal{X}))$  is a presheaf of groups.

Let  $\mathcal{X}$  be an  $H$ -group. Let  $\mathcal{Y}$  be a space.

**Definition 3.3.** The space  $\mathcal{Y}$  is called an  $\mathcal{X}$ -space if there exists a morphism  $a : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$ , such that the following diagram

$$\begin{array}{ccc} \mathcal{X} \times (\mathcal{X} \times \mathcal{Y}) & \xrightarrow{id_{\mathcal{X}} \times a} & \mathcal{X} \times \mathcal{Y} \\ \downarrow a_{\mathcal{X}} \times id_{\mathcal{Y}} & & \downarrow a \\ \mathcal{X} \times \mathcal{Y} & \xrightarrow{a} & \mathcal{Y} \end{array}$$

commutes in  $\mathbf{H}(k)$ .

**Definition 3.4.** Let  $\mathcal{X}$  be an  $H$ -group and let  $\mathcal{Y}$  be an  $\mathcal{X}$ -space.  $\mathcal{Y}$  is called a homogeneous  $\mathcal{X}$ -space if for any essentially smooth henselian  $R$ , the presheaf of groups  $\pi_0^{\mathbb{A}^1}(\mathcal{X})(R)$  acts transitively on  $\pi_0^{\mathbb{A}^1}(\mathcal{Y})(R)$ .

**Remark 3.5.**  $\mathcal{Y}$  is a homogeneous  $\mathcal{X}$ -space if and only if the colimit of the diagram  $\pi_0^{\mathbb{A}^1}(\mathcal{Y}) \xleftarrow{pr} \pi_0^{\mathbb{A}^1}(\mathcal{X}) \times \pi_0^{\mathbb{A}^1}(\mathcal{Y}) \xrightarrow{a} \pi_0^{\mathbb{A}^1}(\mathcal{Y})$  is Nisnevich locally trivial.

**Lemma 3.6.** If

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ A & \longrightarrow & D \end{array}$$

is a homotopy cocartesian square of spaces then, after applying  $a_{Nis}(\pi_0)$ , one gets a cocartesian square of sheaves.

*Proof.* Let  $S \in PSh(Sm/k)$  and let  $\iota(S)$  be the simplicial presheaf such that in every simplicial degree  $k$ ,  $\iota(S)_k = S$ . The face and degeneracy morphisms are identity morphisms. This gives a functor  $\iota : PSh(Sm/k) \rightarrow \Delta^{op}PSh(T)$  which is right adjoint to  $\pi_0$ . Hence  $a_{Nis}(\pi_0)$  also has a right adjoint  $\iota : Sh(Sm/k) \rightarrow \Delta^{op}Sh(Sm/k)$ . This implies,  $a_{Nis}(\pi_0)$  commutes with colimits. Let  $B \xrightarrow{f} A' \xrightarrow{g} A$  be a factorisation of  $B \rightarrow A$ , such that  $f$  is a cofibration and  $g$  is a trivial fibration. Homotopy colimit of the digram  $A \leftarrow B \rightarrow C$  is weakly equivalent to the colimit of  $A' \leftarrow B \rightarrow C$ . As  $a_{Nis}(\pi_0)$  commutes with colimits and  $a_{Nis}(\pi_0(A)) \cong a_{Nis}(\pi_0(A'))$ , we get our result.  $\square$

**Corollary 3.7.** Let  $\mathcal{Y}$  be an  $\mathcal{X}$ -space.  $\mathcal{Y}$  is a homogeneous  $\mathcal{X}$ -space if and only if the homotopy pushout of  $Ex_{\mathbb{A}^1}(\mathcal{Y}) \xleftarrow{pr} Ex_{\mathbb{A}^1}(\mathcal{X}) \times Ex_{\mathbb{A}^1}(\mathcal{Y}) \xrightarrow{a} Ex_{\mathbb{A}^1}(\mathcal{Y})$  is connected.

*Proof.* The proof follows from lemma 3.6 and remark 3.5.  $\square$

**Lemma 3.8.** *Let  $\mathcal{Y}$  be an  $\mathcal{X}$ -space.  $\mathcal{Y}$  is a homogeneous  $\mathcal{X}$ -space if the homotopy pushout of  $\mathcal{Y} \xleftarrow{pr} \mathcal{X} \times \mathcal{Y} \xrightarrow{a} \mathcal{Y}$  is connected.*

*Proof.* By [10, corollary 2.3.22], the canonical morphism  $a_{Nis}(\pi_0(\mathcal{X})) \rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$  (resp.  $a_{Nis}(\pi_0(\mathcal{Y})) \rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))$ ) is surjective as morphism of Nisnevich sheaves. Hence, Nisnevich locally the action of  $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$  on  $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))$  is transitive.  $\square$

**Lemma 3.9.** *Let  $G, G'$  be groups acting on pointed sets  $S, S'$  by action maps  $a, a'$  respectively. Suppose that  $f : G \rightarrow G'$  is a group homomorphism and let  $s : S \rightarrow S'$  be a morphism of pointed sets with trivial kernel such that  $s \circ a = a' \circ (f \times s)$ . If  $G$  acts transitively on  $S$ , then  $s$  is injective.*

*Proof.* Let  $b_S$  (resp.  $b_{S'}$ ) be the base point of  $S$  (resp.  $S'$ ) and let  $a, b \in S$ . Since  $G$  acts transitively on  $S$ , there exist  $g, g' \in G$  such that  $a(g, b_S) = a$  and  $a(g', b_S) = b$ . If  $s(a) = s(b)$ , then  $a'(f(g), b_{S'}) = a'(f(g'), b_{S'})$ . Hence  $a'(f(g^{-1} \cdot g'), b_{S'}) = b_{S'}$ . So  $s(a(g^{-1} \cdot g', b_S)) = b_{S'}$ . But  $s$  is a morphism of pointed sets with trivial kernel, therefore  $a(g^{-1} \cdot g', b_S) = b_S$ . This implies  $a = a(g, b_S) = a(g', b_S) = b$ .  $\square$

Let  $\tilde{Sm}/k$  be the category whose objects are same as objects of  $Sm/k$ , but the morphisms are smooth morphisms. The following argument is taken from [8, Corollary 5.9]

**Lemma 3.10.** *Let  $S$  be a Nisnevich sheaf on  $Sm/k$ . Suppose that for all essentially smooth henselian  $X$ , the map  $S(X) \rightarrow S(K(X))$  is injective. Then  $S(Y) \rightarrow S(K(Y))$  is injective, for all connected  $Y \in Sm/k$ .*

*Proof.* Let  $S'$  be the presheaf on  $\tilde{Sm}/k$ , given by

$$X \in \tilde{Sm}/k \mapsto \prod_i S(K(X_i)),$$

where  $X_i$ 's are the connected components of  $X$ . Then  $S'$  is a Nisnevich sheaf on  $\tilde{Sm}/k$  (as every Nisnevich covering of some  $X \in \tilde{Sm}/k$  splits over some open dense  $U \subset X$ ). The canonical morphism  $S \rightarrow S'$  is injective on Nisnevich stalks. Hence  $S \rightarrow S'$  is sectionwise injective.  $\square$

**Corollary 3.11.** *Let  $S$  be a Nisnevich sheaf on  $Sm/k$ . Suppose that for all essentially smooth henselian  $X$ , the map  $S(X) \rightarrow S(K(X))$  is injective. Then  $S(Y) \rightarrow S(U)$  is injective for any  $Y \in Sm/k$  and any open dense  $U \subset Y$ .*

*Proof.* We can assume that  $Y$  is connected. By lemma 3.10, the morphism  $S(Y) \rightarrow S(K(Y))$  is injective and  $S(U) \rightarrow S(K(Y))$  is injective, hence  $S(Y) \rightarrow S(U)$  is injective.  $\square$

**Lemma 3.12.** *Let  $S$  be a Zariski sheaf on  $Sm/k$ , such that  $S(X) \rightarrow S(U)$  is injective for any  $X \in Sm/k$  and for any open dense  $U \subset X$ . Then  $S$  is  $\mathbb{A}^1$ -invariant if and only if  $S(F) \rightarrow S(\mathbb{A}_F^1)$  is bijective for every finitely generated separable field extension  $F/k$ .*

*Proof.* The only if part is clear. We need to show that for any connected  $X \in Sm/k$ , the morphism  $S(\mathbb{A}_X^1) \rightarrow S(X)$  (induced by the zero section), is bijective. Let

$F := K(X)$ . In the following commutative diagram

$$\begin{array}{ccc} S(\mathbb{A}_X^1) & \longrightarrow & S(X) \\ \downarrow & & \downarrow \\ S(\mathbb{A}_F^1) & \longrightarrow & S(F) \end{array}$$

the left vertical, the right vertical and the bottom horizontal morphisms are injective, thus the top horizontal surjective morphism is injective.  $\square$

We recall the following from [6] and [8, Corollary 5.7]

**Theorem 3.13.** *Let  $X$  be a smooth (or essentially smooth)  $k$ -scheme,  $s \in X$  be a point and  $Z \subset X$  be a closed subscheme of codimension  $d > 0$ . Then there exists an open subscheme  $\Omega \subset X$  containing  $s$  and a closed subscheme  $Z' \subset \Omega$ , of codimension  $d - 1$ , containing  $Z_\Omega := Z \cap \Omega$  and such that for any  $n \in \mathbb{N}$  and for any  $\mathbb{A}^1$ -fibrant space  $\mathcal{X}$ , the map*

$$\pi_n(\mathcal{X}(\Omega/(\Omega - Z_\Omega))) \rightarrow \pi_n(\mathcal{X}(\Omega/(\Omega - Z')))$$

*is the trivial map. In particular, if  $Z$  has codimension 1 and  $X$  is irreducible,  $Z'$  must be  $\Omega$ . Thus for any  $n \in \mathbb{N}$  the map*

$$\pi_n(\mathcal{X}(\Omega/(\Omega - Z_\Omega))) \rightarrow \pi_n(\mathcal{X}(\Omega))$$

*is the trivial map.*

**Remark 3.14** ([8]). *Let  $X$  be an essentially smooth local ring and let  $x$  be the closed point. Let  $U \subset X$  be an open set. We have the following exact sequence of sets and groups for any  $\mathbb{A}^1$ -fibrant space  $\mathcal{X}$  :*

$$\cdots \rightarrow \pi_1(\mathcal{X})(X) \rightarrow \pi_1(\mathcal{X})(U) \rightarrow \pi_0(\mathcal{X})(X/U) \rightarrow \pi_0(\mathcal{X})(X) \rightarrow \pi_0(\mathcal{X})(U)$$

*Applying theorem 3.13 to  $X$  and its closed point  $x$ , we see that  $\Omega = X$  and the morphisms*

$$\pi_n(\mathcal{X})(X/U) \rightarrow \pi_n(\mathcal{X})(X)$$

*are trivial. Hence the morphism of pointed sets*

$$\pi_0(\mathcal{X})(X) \rightarrow \pi_0(\mathcal{X})(U)$$

*has trivial kernel. Taking colimit over open sets, this gives the morphism of pointed sets*

$$\pi_0(\mathcal{X})(X) \rightarrow \pi_0(\mathcal{X})(K(X))$$

*which has trivial kernel. In particular if  $X$  is henselian, then the morphism of pointed sets*

$$a_{Nis}(\pi_0(\mathcal{X}))(X) \rightarrow a_{Nis}(\pi_0(\mathcal{X}))(K(X))$$

*has trivial kernel.*

**Theorem 3.15.** *Let  $\mathcal{X}$  be an  $H$ -group and  $\mathcal{Y}$  be a homogeneous  $\mathcal{X}$ -space. Then  $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$  and  $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))$  are  $\mathbb{A}^1$ -invariant.*

*Proof.* For any connected  $X \in Sm/k$  and any  $x \in X$ , the morphisms of pointed sets

$$\begin{aligned} a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(O_{X,x}^h) &\rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(O_{X,x}^h)) \\ a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(O_{X,x}^h) &\rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(K(O_{X,x}^h)) \end{aligned}$$



have trivial kernel by remark 3.14. By lemma 3.9 and the fact that  $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(O_{X,x}^h)$  is a group, the morphisms mentioned above are injective morphisms of sets. By lemma 3.10, for every  $X \in Sm/k$ , the morphisms

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(X) \rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(X))$$

and

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(X) \rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(K(X))$$

are injective. Hence for any  $X \in Sm/k$  and any open dense subscheme  $U \subset X$ , the morphisms

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(X) \rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(U)$$

and

$$a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(X) \rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{Y}))(U)$$

are injective by corollary 3.11,. Now applying corollary 2.3 and lemma 3.12, we get our result.  $\square$

**Remark 3.16.** *If  $\mathcal{X}$  is an  $H$ -group, then*

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(R) \rightarrow a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(R)$$

*is bijective for any essentially smooth discrete valuation ring  $R$ . Indeed, using remark 3.14 one can easily show that for any essentially smooth discrete valuation ring  $R$ , the group homomorphism*

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(R) \rightarrow \pi_0^{\mathbb{A}^1}(\mathcal{X})(K(R))$$

*is injective. On the other hand, consider the following commutative diagram*

$$\begin{array}{ccc} \pi_0^{\mathbb{A}^1}(\mathcal{X})(R) & \longrightarrow & \pi_0^{\mathbb{A}^1}(\mathcal{X})(K(R)) \\ \downarrow & & \downarrow \wr \\ a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(R) & \longrightarrow & a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(K(R)) \end{array}$$

*where the bottom horizontal morphism is injective by theorem 3.15. The right vertical injective morphism is surjective by lemma 2.8. Hence it is bijective.*

#### 4. APPLICATION AND COMMENTS

By gathering known facts from [11], [9, Theorem 2.4] and [2, Corollary 5.10] one can show that for any connected linear algebraic group  $G$ , such that the almost simple factors of the universal covering (in algebraic group theory sense) of the semisimple part of  $G$  is isotropic and retract  $k$ -rational ([2, Definition 2.2]), the sheaf  $a_{Nis}(\pi_0^{\mathbb{A}^1}(G))$  is  $\mathbb{A}^1$ -invariant. By 3.15, we have the following generalisation.

**Corollary 4.1.** *Let  $G$  be any sheaf of groups on  $Sm/k$  and  $B$  be any subsheaf of groups. Then  $a_{Nis}(\pi_0^{\mathbb{A}^1}(G))$  is  $\mathbb{A}^1$ -invariant and  $a_{Nis}(\pi_0^{\mathbb{A}^1}(G/B))$  is  $\mathbb{A}^1$ -invariant. Here  $G/B$  is the quotient sheaf in Nisnevich topology.*

We recall from [8, Definition 7] the following definition.

**Definition 4.2.** *A sheaf of groups  $G$  on  $Sm/k$  is called strongly  $\mathbb{A}^1$ -invariant if for any  $X \in Sm/k$ , the map*

$$H_{Nis}^i(X, G) \rightarrow H_{Nis}^i(\mathbb{A}_X^1, G)$$

*induced by the projection  $\mathbb{A}_X^1 \rightarrow X$ , is bijective for  $i \in \{0, 1\}$ .*

Let  $\mathcal{X}$  be a pointed space. By [8, Theorem 9], for any pointed simplicial presheaf  $\mathcal{X}$ , the sheaf of groups  $a_{Nis}(\pi_0(\Omega(Ex_{\mathbb{A}^1}(\mathcal{X})))) = \pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$  is strongly  $\mathbb{A}^1$ -invariant. Here  $x$  is the base point of  $\mathcal{X}$  and  $\Omega(Ex_{\mathbb{A}^1}(\mathcal{X}))$  is the loop space of  $Ex_{\mathbb{A}^1}(\mathcal{X})$ . So for any space  $\mathcal{X}$ , which is the loop space of some  $\mathbb{A}^1$ -local space  $\mathcal{Y}$ , [8, Theorem 9] gives the  $\mathbb{A}^1$ -invariance property for  $a_{Nis}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))$ . We end this section by showing that there exists an  $\mathbb{A}^1$ -local  $H$ -group which is not a loop space of some  $\mathbb{A}^1$ -local space. This will imply that the statement of the theorem 3.15 for  $H$ -groups is not a direct consequence of [8, Theorem 9]. It is enough to show that there exists sheaf of groups  $G$  which is  $\mathbb{A}^1$ -invariant, but not strongly  $\mathbb{A}^1$ -invariant.

Let  $\mathbb{Z}[\mathbb{G}_m]$  be the free presheaf of abelian groups generated by  $\mathbb{G}_m$ .

**Remark 4.3.** For any  $X \in Sm/k$  and a dominant morphism  $U \rightarrow X$ , the canonical morphism  $\mathbb{Z}[\mathbb{G}_m](X) \rightarrow \mathbb{Z}[\mathbb{G}_m](U)$  is injective. Indeed, any nonzero  $a \in \mathbb{Z}[\mathbb{G}_m](X)$  can be written as  $a = \sum_{i=1}^n a_i \cdot g_i$ , where  $g_i \in \mathbb{G}_m(X)$  and  $a_i \in \mathbb{Z} \setminus \{0\}$  such that  $g_i \neq g_{i'}$  for  $i \neq i'$ . Suppose  $a|_U = 0$ , i.e.,  $\sum_{i=1}^n a_i \cdot g_i|_U = 0$ . Since  $\mathbb{G}_m(X) \rightarrow \mathbb{G}_m(U)$  is injective,  $g_i|_U \neq g_{i'}|_U$  for  $i \neq i'$ . This implies  $a_i = 0$  for all  $i$ . Hence  $a = 0$ .

The presheaf  $\mathbb{Z}[\mathbb{G}_m]$  is not a Nisnevich sheaf. But it is not far from being a Nisnevich sheaf.

**Lemma 4.4.** The Nisnevich sheafification  $a_{Nis}(\mathbb{Z}[\mathbb{G}_m])$  is the presheaf that associates to every smooth  $k$ -scheme  $X = \coprod_i X_i$ , the abelian group  $\prod_i \mathbb{Z}[\mathbb{G}_m](X_i)$ , where  $X_i$ 's are the connected components of  $X$ .

*Proof.* Let  $\mathcal{F}$  be the presheaf that associates to every smooth  $k$ -scheme  $X = \coprod_i X_i$ , the abelian group  $\prod_i \mathbb{Z}[\mathbb{G}_m](X_i)$ , where  $X_i$ 's are the connected components of  $X$ . It is enough to prove that  $\mathcal{F}$  is a Nisnevich sheaf. We need to show that for any elementary distinguished square in  $Sm/k$

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X, \end{array}$$

the induced commutative square

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(W) \end{array}$$

is cartesian. By the construction of  $\mathcal{F}$  we can assume that  $X, W, V, U$  are connected. So, it is enough to prove that

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{G}_m](X) & \longrightarrow & \mathbb{Z}[\mathbb{G}_m](V) \\ \downarrow & & \downarrow \\ \mathbb{Z}[\mathbb{G}_m](U) & \longrightarrow & \mathbb{Z}[\mathbb{G}_m](W) \end{array}$$

is cartesian. Let  $a \in \mathbb{Z}[\mathbb{G}_m](U)$  and let  $b \in \mathbb{Z}[\mathbb{G}_m](V)$  such that  $a|_W = b|_W$ . We can write  $a = \sum_{i=1}^n a_i \cdot f_i$  and  $b = \sum_{j=1}^m b_j \cdot g_j$ , where  $a_i, b_j \in \mathbb{Z} \setminus \{0\}$  and  $(f_i, g_j) \in \mathbb{G}_m(U) \times \mathbb{G}_m(V)$  such that  $f_i \neq f_{i'}$  and  $g_j \neq g_{j'}$  for all  $i \neq i'$  and  $j \neq j'$ .

Since all the morphisms are dominant,  $g_j|_W \neq g_{j'}|_W$  and  $f_i|_W \neq f_{i'}|_W$  for all  $i \neq i'$  and  $j \neq j'$ . Hence, for every  $i$  there exists at most one  $j$  such that  $f_i|_W = g_j|_W$ . Suppose for some  $f_{i'}$ ,  $f_{i'}|_W \neq g_j|_W$  for all  $j$ . Then we can write

$$\left(\sum_{i=1}^n a_i \cdot f_i|_W\right) - \left(\sum_{j=1}^m b_j \cdot g_j|_W\right) = a_{i'} f_{i'} + \sum_{k=1}^l c_k \cdot h_k = 0,$$

where  $h_k \neq h_{k'}$  for all  $k \neq k'$  and  $f_{i'} \neq h_k$  for all  $k$ . This implies  $a_{i'} = 0$ , which gives a contradiction. Hence, for every  $i$  there exists exactly one  $j$  such that  $f_i|_W = g_j|_W$ . Therefore,  $m = n$ . Also we can write  $a = \sum_{i=1}^n a'_i \cdot f'_i$ , such that  $a'_i = b_i$  and  $f'_i|_W = g_i|_W$ . Since  $\mathbb{G}_m$  is a Nisnevich sheaf, we get  $g'_i \in \mathbb{G}_m(X)$  which restricts to  $f'_i$  and  $g_i$ . This gives a section  $c = \sum_{i=1}^n b_i \cdot g'_i \in \mathbb{Z}[\mathbb{G}_m](X)$  which restricts to  $a$  and  $b$ . The uniqueness of  $c$  follows from the remark 4.3.  $\square$

As  $\mathbb{G}_m$  is pointed by 1,  $a_{Nis}(\mathbb{Z}[\mathbb{G}_m]) \cong \mathbb{Z} \oplus \mathbb{Z}(\mathbb{G}_m)$ . Here  $\mathbb{Z}$  is the sheaf generated by the point 1. Let  $A$  be a sheaf of abelian groups on  $Sm/k$ . To give a morphism  $\mathbb{G}_m \rightarrow A$ , such that 1 gets mapped to  $0 \in A$ , is equivalent to give a morphism  $\mathbb{Z}(\mathbb{G}_m) \rightarrow A$  of abelian sheaves. Since  $\mathbb{G}_m$  is  $\mathbb{A}^1$ -invariant,  $a_{Nis}(\mathbb{Z}[\mathbb{G}_m])$  is  $\mathbb{A}^1$ -invariant. This implies  $\mathbb{Z}(\mathbb{G}_m)$  is  $\mathbb{A}^1$ -invariant.

**Remark 4.5.** Let  $\sigma_1 : \mathbb{G}_m \rightarrow \underline{K}_1^{MW}$  be the canonical pointed morphism (see [8, page 86]). For any finitely generated separable field extension  $F/k$ , the morphism maps  $u \in F^*$  to the corresponding symbol  $[u] \in K_1^{MW}(F)$ . The induced morphism  $\mathbb{Z}(\mathbb{G}_m) \rightarrow \underline{K}_1^{MW}$  is not injective. Indeed, we can choose  $u \in F^* \setminus 1$  such that  $u(u-1)$  is not 1. The element  $[u(u-1)] - [u] - [u-1]$  is zero in  $K_1^{MW}(F)$ , but it is non zero in  $\mathbb{Z}(\mathbb{G}_m)(F)$ .

**Lemma 4.6.** The  $\mathbb{A}^1$ -invariant sheaf of abelian groups  $\mathbb{Z}(\mathbb{G}_m)$  is not strongly  $\mathbb{A}^1$ -invariant.

*Proof.* Suppose  $\mathbb{Z}(\mathbb{G}_m)$  is strongly  $\mathbb{A}^1$ -invariant. Then by [8, Theorem 2.37], the morphism  $id : \mathbb{Z}(\mathbb{G}_m) \rightarrow \mathbb{Z}(\mathbb{G}_m)$  can be written as  $\phi \circ \sigma_1$  for some unique  $\phi$ . This implies  $\sigma_1$  is injective which contradicts remark 4.5.  $\square$

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